MOMENTS OF THE ABSORPTION TIME IN A FINITE MARKOV CHAIN

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Abstract

If all recurrent states are included in a closed subset $E_J = \bigcup_{j \in J} E_j$ of states in a finite Markov chain then the probability

$$F_{iJ}(t) = \sum_{j \in J} p_{ij}^{(t)} = \sum_{k} p_{ik}^{(1)} F_{kJ}(t-1)$$

represents the cumulative probability distribution of a random variable T_{iJ}, the number of steps required for a chain which starts in E_i to first reach E_J. The moments of T_{iJ} may be obtained directly as linear functions of the one-step transition probabilities, as

$$\mathcal{E}(T_{iJ}) - 1 = \sum_{k \notin J} p_{ik}^{(1)} \mathcal{E}(T_{kJ})$$

$$\mathcal{E}(T_{iJ}^2) - 2\mathcal{E}(T_{iJ}) + 1 = \sum_{k \notin J} p_{ik}^{(1)} \mathcal{E}(T_{kJ}^2) \quad .$$

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If $\|p_{ij}^{(t)}\| = \|p_{ij}^{(1)}\|^t$ is the t-step transition probability matrix of a Markov process on the states $E_1,\cdots,E_n$ then the probability $F_{iJ}(t)$ that a chain which starts in E_i will be absorbed into a given closed set of states

$$E_J = \bigcup_{j\in J} E_j$$

on or before the t^{th} step is

$$(1) \qquad F_{iJ}(t) = \sum_{j\in J} p_{ij}^{(t)} = \sum_{k} p_{ik}^{(1)} F_{kJ}(t-1) \quad .$$

When all recurrent states are included in E_J then $F_{iJ}(\infty) = 1$, and if T_{iJ} is the time required to first achieve E_J starting from $E_i \notin E_J$ then

$$P(T_{iJ} \le t) = F_{iJ}(t)$$

so

$$f_{iJ}(t) \underset{def}{\equiv} P(T_{iJ}=t) = F_{iJ}(t) - F_{iJ}(t-1) \approx \frac{dF_{iJ}(t)}{dt}$$

Modal values and percentiles of the T_{iJ} are thus readily computed once the functions $F_{iJ}(t)$ are determined by either method indicated in (1). Computation of the moments of these distributions, however, does not require prior calculation of the distribution functions themselves; moments may be obtained directly as linear functions of the one-step transition probabilities $p_{ij}^{(1)}$.

Multiplying both sides of the equation

$$f_{iJ}(t+1) = \sum_{k \notin J} p_{ik}^{(1)} f_{kJ}(t)$$

by t and summing gives

$$\mathcal{E}_J(T_{iJ}) - 1 = \sum_{k \notin J} p_{ik}^{(1)} \mathcal{E}_J(T_{kJ}) \quad .$$

Similarly, for the second moments,

$$\sum_{t=1}^{\infty} t^2 f_{iJ}(t+1) = \sum_{k \notin J} p_{ik}^{(1)} \sum_{t=1}^{\infty} t^2 f_{kJ}(t)$$

or

$$\mathcal{E}_J(T_{iJ}^2) - 2\mathcal{E}_J(T_{iJ}) + 1 = \sum_{k \notin J} p_{ik}^{(1)} \mathcal{E}_J(T_{kJ}^2)$$

and so on to consecutively higher momemts.

Note that if E_J partitions into a collection of (disjoint) closed sets, $E_J = \{E_{J_1}, \cdots, E_{J_k}\}$ then the same methods may be employed to find the conditional moments of T_{iJ_h}, the time to absorption into E_{J_h}, given that the process is ultimately abosrbed into E_{J_h}. The only added complication is that

$$\sum_{t=1}^{\infty} t^r [F_{iJ_h}(t) - F_{iJ_h}(t-1)] = F_{iJ_h}(\infty)\mathcal{E}_{J_h}(T^r_{iJ_h})$$

where

$$F_{iJ_h}(t) = \sum_{j\in J_h} p_{ij}^{(t)} = \sum_{j\notin J} p_{ij}^{(1)} F_{jJ_h}(t-1) + \sum_{j\in J_h} p_{ij}^{(1)} \;.$$

Thus, the solution to an additional set of linear equations, for the case r=0, must precede the calculation of the higher moments.